

# SYMPLECTIC QUANTUM MECHANICS AND CHERN-SIMONS GAUGE THEORY II: MAPPING TORI OF TORI

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ABSTRACT. We compute the semiclassical formulas for the partition functions obtained using two different Lagrangians: the Chern-Simons functional and the symplectic action functional.

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## 1. INTRODUCTION

**1.1. Chern-Simons gauge theory.** This article is a companion to [20]. It treats Chern-Simons gauge theory and a second theory, symplectic quantum mechanics (SQM), a field theory in  $0 + 1$  dimensions for which the Lagrangian is the symplectic action functional. This paper treats the special case of mapping tori of tori. We shall introduce a Lie group  $G$

(usually  $G = SU(2)$ ) and an integer parameter  $k$  (the level). The background for this paper is as in §1 of [20].

**1.2. Summary of results.** Our work treats the Chern-Simons partition function for mapping tori  $\Sigma_\beta$  of surfaces  $\Sigma$ . In this article we treat only the case when  $\Sigma$  is a 2-torus. The main aim here is to use the reciprocity formula for Gauss sums to demonstrate the large  $k$  limit (??) of the Chern-Simons partition function for mapping tori of tori. As a complement to [20], we also compute the SQM partition function explicitly and show the terms in it agree with those in the rigorous large  $k$  limit of the Chern-Simons partition function. In this case, a formal argument shows the stationary phase approximation to the Chern-Simons partition function is exact (i.e., the action is precisely quadratic.) This is confirmed by our rigorous calculations in [19], recalled in §5.

Here we restrict the explicit treatment of symplectic quantum mechanics to mapping tori of tori. When  $\Sigma$  is a torus, the stationary phase approximation to the symplectic quantum mechanics path integral is indeed exact. We may thus expect exact agreement with the value for the partition function, rather than agreement only with the leading term in an asymptotic expansion. We present two computations. For  $G = SU(2)$ , we have treated an arbitrary element  $U$  of  $SL(2, \mathbb{Z})$ . For arbitrary classical groups  $G$ , we have for simplicity treated only  $U = T^p S$  (see (11) and (12) below).

The notation in this paper is as in [20]. The remainder of this article is organized as follows. In §2 we specialize to mapping tori of tori. We prove (formally) the exactness of stationary phase (Proposition 2.4), and explicitly evaluate the quantities appearing in the stationary phase approximation of SQM.

§3 computes the SQM partition function. In §4, we discuss the natural framings of mapping tori of tori, and how they enter in the comparison of the Chern-Simons and SQM partition functions. §5 recalls results from [19] which give the large  $k$  limit of the Chern-Simons partition function using Gauss sums. In fact this calculation demonstrates that the large  $k$  expression gives the exact value, in accordance with our formal argument in §2.

The main result in this article is the comparison of the semiclassical formula for SQM with the semiclassical formula for Chern-Simons gauge theory of a mapping torus of a torus (the critical points of both Lagrangians correspond to fixed points of the diffeomorphism defining the mapping torus) – see Proposition 3.3 and (19).

*Remark:* Much of the material in this article derives from the author's D. Phil. thesis [18]. Some other results from this thesis have already been published in [19].

## 2. MAPPING TORI OF TORI

**2.1. General method.** The case of mapping tori of tori is simpler in many respects than the general case, as described in the introduction. Examples of three-manifolds  $\Sigma_\beta$  for  $\Sigma$  a torus may be obtained by 0-surgery on fibred knots whose fibre is a punctured torus: some of these are discussed in [21].

One difficulty with the torus case is that the moduli space  $\mathcal{M}(\Sigma)$  is singular, as is obvious from its description as a quotient space  $(T \times T)/W$ . Some of the fixed points of the symplectic diffeomorphism  $f$  of  $T \times T/W$  are in fact singular points, notably the product connection. It is not obvious how to rigorously define the quantities  $\det D_x$  and  $\text{Tr } \tilde{f}_x$  appearing in the

contribution to the SQM partition function from such fixed points  $x$ . These points are points fixed by some nontrivial  $w \in W$ : for  $SU(2)$  they are the central flat connections.

We shall circumvent this difficulty by working on  $T \times T$ . Observe that if  $A \in T \times T$  is a fixed point of a linear map  $U$  acting on  $T \times T/W$ , then there is some  $w \in W$  such that  $A$  is a fixed point of  $wU$  acting on  $T \times T$ . The SQM data  $\det D_A$  and  $\text{Tr } \tilde{f}_A$  for the map  $f = U$  on  $T \times T/W$  thus naturally correspond to the SQM data for  $f = wU$  on  $T \times T$ . (Here,  $D_A$  is the operator  $-\frac{1}{2}J(\frac{d}{dt} + E)$  defined in [20]) It is thus natural to consider all such maps  $f = wU$  on  $T \times T$  and sum their SQM partition functions. In fact,  $A$  solves  $wUA = A$  in  $T \times T$  if and only if  $w'A$  solves  $w'w(w')^{-1}U(w'A) = w'A$ , so it is necessary to divide our fixed point sum by  $|W|$ .

This prescription determines  $\text{Tr } \tilde{f}_A$  and  $\det D_A$  for  $A \in T \times T$ , provided  $A$  is fixed by only one of the maps  $wU$ , i.e., provided no nontrivial element of  $W$  fixes  $A$ . For those elements  $A$  which are fixed by some element in  $W$ , we sum the contributions to the SQM partition functions for all  $wU$  that fix  $A$ .

The regularization procedure for the eta invariant for a general moduli space (see [20]) enables us to replace the difference of eta invariants  $\eta(A_+) - \eta(A_-)$  by a shift in the coefficient of the symplectic action functional  $S(A_+) - S(A_-)$  from  $k$  to  $k + h$ , plus a term involving the spectral flow mod 4 of the path of operators between  $D_{A_+}$  and  $D_{A_-}$  associated to the gradient of the symplectic action functional. If  $A_+$  and  $A_- \in T \times T$  are fixed by  $w_+U$ ,  $w_-U$  where  $w_+$  and  $w_-$  are *different*, we cannot naturally define the spectral flow between  $D_{A_+}$  and  $D_{A_-}$ . We can, however, always define the spectral flow of the family of operators associated to the gradient of the symplectic action functional for  $f = wU$  from a fixed point  $A$  of  $f$  to the product connection  $A_0$ , since  $A_0$  is fixed by  $wU$  for all  $w$ . In fact, it turns out (see Proposition 2.7) that this spectral flow is zero. We thus just need an ansatz to replace the spectral flow between the operator  $D_{A_0}(wU)$  and the operator  $D_{A_0}(U)$ . This ansatz (based on the results of our rigorous calculations using Gauss sums) is

$$(1) \quad SF(D_{A_0}(U), D_{A_0}(wU)) = 1 - \det w \pmod{4}.$$

This gives the formula

$$(2) \quad Z_{SQM}(U, k) = i^\mu \frac{1}{||W||} \sum_{w \in W} \sum_{\substack{A \in T \times T: \\ wUA=A}} \det w \frac{\text{Tr } \tilde{f}_A^{k+h}}{||\det D_A(wU)||^{1/2}}.$$

Here,  $\mu(A_0)$  is the “defect” resulting from a certain integer choice. This is (5.11) in [19], which requires Conjecture 5.8 in that paper. This conjecture has been proved by Himpel in [17].

**Remark 2.1.** Even without the factor  $i^\mu$ , the overall sign of the SQM partition function (2) is ambiguous: if  $-1 \in W$ , then under replacement of  $U$  by  $-U$ , the SQM partition function changes by  $\det w$ , which is 1 or  $-1$ .

2.1.1. *Regularization of eta invariant: the torus case.* In regularizing the eta invariant in SQM for a symplectic manifold  $N$ , we obtained a correction term  $-2(\frac{i}{2\pi} \int_{\tilde{u}} \text{Tr } F_{\nabla})$ , where  $\text{Tr } F_{\nabla}$  was the curvature of the canonical bundle  $\mathcal{K}$  of  $N$ , viewed as a bundle over the mapping

torus  $N_f$ . We replaced this by  $2 \int_u \alpha$ , where  $\alpha$  was a 2-form on  $N$  representing the cohomology class  $c_1(\mathcal{K})$ .

In the case when  $N = T \times T/W$ , one must make sense of the canonical bundle of  $T \times T/W$  rather than using the canonical bundle of  $T$  (which is of course trivial). Observe that for a branched covering  $N \xrightarrow{\pi} M$  of complex manifolds, we have the ‘‘Hurwitz formula’’

$$(3) \quad \pi^* c_1 \mathcal{K}_M + D = c_1 \mathcal{K}_N,$$

where  $D$  is the divisor corresponding to the branch locus (taken with some multiplicity). For us,  $N = T \times T$ , and  $M = T \times T/W$  is no longer a manifold. Nonetheless, we adopt this as a definition of  $\pi^* c_1(\mathcal{K}_M)$ ; this yields (see [8], (5.30))

$$\pi^* c_1 \mathcal{K}_{T \times T/W} = 2h \left( \frac{\omega}{2\pi} \right),$$

where  $\omega$  is the basic symplectic form. This leads as before to the shift of the coefficient of the symplectic action functional from  $k$  to  $k + h$ .

**2.2. Lifting  $f$  to the prequantum line bundle.** We now discuss the SQM data for  $\mathcal{M}$  when  $\Sigma$  is a torus. We may view  $\mathfrak{t} \oplus \mathfrak{t}$  as a subspace of the space of connections  $\mathcal{A}$  on  $\Sigma$ , and the actions of  $W$  and  $\Lambda = \Lambda^R \oplus \Lambda^R$  as gauge transformations. It is easy to check by explicit calculation that our lifting of these actions to  $\mathcal{L}$  coincides with the lifting via the Chern-Simons functional described in [20] (see (68) in that paper).

**Notation:** The diffeomorphism  $\beta$  of  $\Sigma$  corresponds to an element  $U \in SL(2, \mathbb{Z})$ . We shall write  $f$  or  $f_U$  for the corresponding map on  $T \times T/W$ , and  $\tilde{f}$  or  $\tilde{f}_U$  for its lift to the prequantum line bundle  $\mathcal{L}$  over  $T \times T/W$  or  $T \times T$ .

We now choose a lift of  $f_U : \mathcal{M} \rightarrow \mathcal{M}$  to  $\mathcal{L}$ , preserving the connection. We choose the trivial lift to the trivial bundle over  $\mathbb{A}$ :

$$(4) \quad \tilde{f}_U(A, z) = (UA, z)$$

This is easily shown to preserve the connection on  $\mathcal{L}$ . However, all lifts to the prequantum line bundle preserving the connection coincide up to a constant in  $U(1)$ . That the lift (4) coincides *precisely* with the lift using the Chern-Simons functional follows from the fact that they agree on the product connection  $A = 0$ .

We need to choose a lift of  $\beta : \Sigma \rightarrow \Sigma$  to  $\tilde{\beta} : P_\Sigma \rightarrow P_\Sigma$ , and a flat connection  $A_0$  preserved by  $\tilde{\beta}$ . We do this by choosing a trivialization of  $P_\Sigma$  and letting  $A_0$  be the product connection and  $\tilde{\beta}$  the trivial lift. This choice of  $\tilde{\beta}$  then preserves the subspace  $\mathbb{A}$  of connections with constant coefficients in  $\mathfrak{t}$ . We identify  $A_0$  with  $0 \in \mathfrak{t}$ ; this enables us to lift the action of  $\tilde{\beta}$  on  $T \times T$  to the linear action of  $U$  on  $\mathbb{A}$ . Of course the connection on the symplectic affine space  $\mathbb{A}$  is simply the restriction to  $\mathbb{A}$  of the connection defined in [20].

We now show

**Lemma 2.2.** *The lift of  $U \in SL(2, \mathbb{Z})$  given by (4) and our lift of  $w \in W$  are equivariant with respect to the action of the lattice  $\Lambda$  on  $\mathcal{L}$ .*

**Proof:** Write  $V$  for the corresponding linear maps on  $\mathfrak{t} \oplus \mathfrak{t}$ . The equivariance condition is characterized by the following equation on  $\mathcal{L} = \mathbb{A} \times \mathbb{C}$ :

$$(VA + V\lambda, e_\lambda(A)v) = (VA + V\lambda, e_{V(\lambda)}(VA)v).$$

Now

$$\frac{e_{V(\lambda)}(VA)}{e_\lambda(A)} = \frac{\epsilon(V\lambda)}{\epsilon(\lambda)},$$

so we need  $\epsilon(V\lambda) = \epsilon(\lambda)$ . Actually we need only check this for  $\lambda$  in some basis of lattice vectors.

We fix the coroot basis  $\{h_\alpha\}$  of  $\Lambda^R$ , and correspondingly a basis  $\{h_\alpha^{(1)}, h_\alpha^{(2)}\}$  of  $\Lambda^R \oplus \Lambda^R$ . We define the theta-characteristic by

$$(5) \quad \epsilon(h_\alpha^{(i)}) = 1.$$

Then for  $U \in SL(2, \mathbb{Z})$ ,  $Uh_\alpha^{(i)} = mh_\alpha^{(1)} + nh_\alpha^{(2)}$  for some  $m, n \in \mathbb{Z}$ ; because  $h_\alpha$  does not mix with the other coroots  $h_\beta, \beta \neq \alpha$  under  $U$  and because  $\langle h_\alpha, h_\alpha \rangle \in 2\mathbb{Z}$ , we have  $\epsilon(Uh_\alpha^{(i)}) = 1$ .

Similarly for  $w \in W$ ,  $wh_\alpha^{(i)} = \sum_\beta n_\beta h_\beta^{(i)}$ : the two summands  $\mathbf{t}_1$  and  $\mathbf{t}_2$  in  $\mathbf{t} \oplus \mathbf{t}$  do not mix. So since  $\omega$  pairs  $\mathbf{t}_1$  with  $\mathbf{t}_2$ , again  $\epsilon(wh_\alpha^{(i)}) = 1$ . Hence the definition (4) does indeed give a lift to  $\mathcal{L}$ .  $\square$

### Remark 2.3. : Theta-characteristics

The choice of a theta characteristic for a bundle  $\mathcal{L}$  on  $T \times T$  is the specification of  $w_1(\mathcal{L}) \in H^1(T \times T, \mathbb{Z}_2)$ . We know that  $(T \times T)/W$  is simply connected, so bundles with different choices of theta-characteristic on  $T \times T$  descend to isomorphic bundles on  $T \times T/W$ , and the choice of theta-characteristic is irrelevant for our purposes. For convenience in specifying the lift of  $SL(2, \mathbb{Z})$  to  $\mathcal{L}$ , we make the particular choice (5) for the theta characteristic. A different choice would force us to choose a different lift in order to make it equivariant with respect to the  $\Lambda$  action.

The theta characteristic we have chosen is obviously identically 1 in the  $SU(2)$  case.  $\square$

#### 2.2.1. Stationary phase approximation.

**Proposition 2.4.** *The stationary phase approximation for the SQM partition function corresponding to the moduli space of flat connections on a torus is exact.*

**“Proof” (formal):** We view  $T \times T$  as  $(\mathbf{t} \oplus \mathbf{t})/(\Lambda^R \oplus \Lambda^R)$ . A basis of  $\mathbf{t}$  then defines coordinates on  $T \times T$ , in which the symplectic form  $\omega$  on  $\mathbf{t} \oplus \mathbf{t}$  is a 2-form with constant coefficients and the diffeomorphism  $f$  is a linear map. The Lagrangian is defined by parallel transport in  $\mathcal{L}$  around a path  $\gamma$  with  $f(\gamma(0)) = \gamma(1)$ . Near a critical point  $x_0$  this is given by

$$L(\gamma) = \int_u \omega$$

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where  $u(t, \tau) : \mathbb{R} \times I \rightarrow N$  is a homotopy from the constant path  $\gamma_{x_0}$  to  $\gamma$  with  $u(t+1, \tau) = f(u(t, \tau))$ . Because  $f$  is linear, we may take  $u(t, \tau) = \tau\gamma(t)$ . Because  $\omega$  has constant coefficients, the integral becomes

$$\int_{I \times I} \tau \omega(\gamma, \dot{\gamma}) dt d\tau = \frac{1}{2} \int_I \omega(\gamma, \dot{\gamma}) dt,$$

which is precisely quadratic: hence the stationary phase approximation is exact.  $\square$

**2.3. Fixed points of  $f$ , and action at the fixed points.** If  $A$  is a fixed point of  $f$  on  $T \times T/W$ , there are  $w \in W$  and  $\lambda \in \Lambda$  such that

$$(6) \quad wUA - A = \lambda$$

in  $\mathfrak{t} \oplus \mathfrak{t}$ . The trace of  $\tilde{f}$  at a fixed point is computed as follows:

$$\begin{aligned} \widetilde{wf}(A, v) &= (wU(A), v) = (A + \lambda, v) \\ &= (A, e_\lambda(A)^{-1}v), \end{aligned}$$

In other words

$$\begin{aligned} (7) \quad \text{Trace } \tilde{f}|_A &= e_\lambda(A)^{-1} \\ &= \exp \frac{i}{2} k \omega(A, \lambda) \epsilon^k(\lambda). \end{aligned}$$

Explicitly,

**Lemma 2.5.** *The fixed points  $A_\lambda$  are in correspondence with  $\lambda = (\lambda_1, \lambda_2) \in \Lambda/(wU - 1)\Lambda$ : we define*

$$A_\lambda = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = (wU - 1)^{-1} \lambda.$$

Furthermore, the trace of the lift  $\tilde{f}$  at the fixed point  $A_\lambda$  is given by

$$\begin{aligned} (8) \quad \text{Trace } \tilde{f}|_{A_\lambda} &= \exp \frac{ik}{2} \omega([wU - 1]^{-1} \lambda, \lambda) \epsilon^k(\lambda), \\ &= \exp(-ik\pi \langle [wU - 1]^{-1} \lambda, S\lambda \rangle) \epsilon^k(\lambda) \end{aligned}$$

where the theta-characteristic  $\epsilon(\lambda)$  is defined by (5).  $\square$

By the discussion after (4), we actually have that

$$(9) \quad \text{Trace } \tilde{f}|_A = e^{2\pi i CS(\tilde{A})},$$

where  $\tilde{A}$  is the flat connection on  $\Sigma_\beta$  corresponding to  $A$ . This proves:

**Theorem 2.6.** *The Chern-Simons invariant of the flat connection  $A_\lambda$  on the mapping torus  $\Sigma_U$  of the torus  $\Sigma$  is:*

$$CS(A_\lambda) = \frac{1}{4\pi} \omega((wU - 1)^{-1} \lambda, \lambda) + \begin{cases} 0, & \text{if } \epsilon(\lambda) = 1; \\ \frac{1}{2}, & \text{if } \epsilon(\lambda) = -1. \end{cases} \quad \square$$

Kirk and Klassen ([21], Th. 5.6) have obtained this result for  $G = SU(2)$ .

**2.4. Absolute value of determinant.** As discussed in [20], for the SQM operator  $D$ , the value of  $|\det D|^{-1/2}$  is  $|\det(df - 1)|^{-1/2}$ , where  $df : T_x M \rightarrow T_x M$  at the fixed point  $x$  of  $f$ . For  $M = T \times T/W$  this becomes

$$(10) \quad |\det D|^{-1/2} = |\det(wU - 1)|^{-1/2},$$

for  $w, U$  acting on  $\mathbb{A}$ .

**2.5. Spectral flow.** Recall that  $D$  was  $-\frac{J}{2}(d/dt + E)$ , where  $E$  was chosen in  $\mathfrak{sp}(2n) \otimes \mathbb{C}$  such that  $\exp E = wf$ . Above (1), we have discussed the ansatz to make sense of the spectral flow between fixed points of  $U$  and fixed points of  $wU$ , when  $w \neq 1$ . Here we show the following:

**Proposition 2.7.** *Consider the spectral flow of the operator  $\mathcal{D}$  corresponding to the gradient of the symplectic action functional. Consider a fixed value of  $w \in W$ . Between two fixed points  $x_+, x_-$  of  $wU$  in  $T \times T$ , this spectral flow is 0.*

**Proof:** The spectral flow between  $x_+$  and  $x_-$  is the difference of Maslov indices  $\mu(x_+) - \mu(x_-)$ . The Maslov index  $\mu(\Psi)$  is associated to a path  $\Psi$  in  $Sp(2n, \mathbb{R})$ . Appropriate paths  $\Psi_\pm$  are obtained from a trivialization of the tangent bundle (satisfying appropriate periodicity conditions) over a strip joining the two fixed points  $x_+$  and  $x_-$ . (For details see [10] after equation (4.4).) Using the canonical trivialization of the tangent bundle of  $T \times T$  and a path of linear symplectic maps  $f_t \in SL(2, \mathbb{R})$  joining 1 and  $U$ , one easily sees that a trivialization may be constructed so that the paths  $\Psi_\pm$  are the same. Thus the associated Maslov indices are the same.  $\square$

### 3. SQM PARTITION FUNCTION

**3.1.  $G = SU(2)$ .** We shall present the calculation of the SQM partition function for  $T \times T/W$ . We denote by  $U$  an arbitrary element

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of  $SL(2, \mathbb{Z})$  (provided  $U$  is not parabolic, i.e.,  $\text{Tr}(U) \neq \pm 2$ ).

We wish to compute the SQM formula (2) for the mapping torus partition function. From (10), we need the quantity

$$\det(wU - 1) = 2 \mp (a + d).$$

Equation (8) allows us to evaluate the action:

$$(wU - 1)^{-1} = -\frac{1}{a + d \mp 2} \begin{bmatrix} d \mp 1 & -b \\ -c & a \mp 1 \end{bmatrix},$$

$$\begin{aligned} L(A_\lambda) &= -\pi \langle (wU - 1)^{-1} \lambda, S \lambda \rangle \\ &= -\pi \langle \lambda, (wU^t - 1)^{-1} S \lambda \rangle, \end{aligned}$$

where

$$(11) \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL(2, \mathbb{Z})$$

and

$$(12) \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in SL(2, \mathbb{Z})$$

descend to generators of the group  $PSL(2, \mathbb{Z})$ . (Here,  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathfrak{t} \oplus \mathfrak{t}$ , and  $S \in SL(2, \mathbb{Z})$  acts on  $\mathfrak{t} \oplus \mathfrak{t}$ .) For  $SU(2)$  this becomes

$$(13) \quad L(A_\lambda) = \frac{2\pi}{a + d \mp 2} (-c\lambda_1^2 + b\lambda_2^2 + (a - d)\lambda_1\lambda_2),$$

where we have used the coroot basis and the inner product to identify  $\Lambda^R$  with  $\mathbb{Z}$ . Our expression then reads

$$(14) \quad \begin{aligned} Z &= \frac{1}{2} i^\mu \sum_{\pm} \sum_{\lambda} \pm \frac{1}{\sqrt{|a + d \mp 2|}} \times \\ &\times \exp \left\{ 2\pi i \frac{(k + h)}{a + d \mp 2} (-c\lambda_1^2 + b\lambda_2^2 + (a - d)\lambda_1\lambda_2) \right\}, \end{aligned}$$

where we sum over  $\lambda = (\lambda_1, \lambda_2) \in \Lambda/(\pm U - 1)\Lambda$ .

**3.2. General  $G$ .** In the case of general  $G$  we shall restrict ourselves for simplicity to one specific family, namely those  $U \in SL(2, \mathbb{Z})$  for which  $c = 1$ .

**Notation:**  $p$  will denote  $\text{Tr}(U)$ .

**Lemma 3.1.** *We have*

$$|\det(w \otimes U - 1)| = |\det(\text{Tr}(U) - w - w^{-1})|.$$

**Proof:** If the eigenvalues of  $U$  are  $\lambda, \lambda^{-1}$  and those of  $w$  are  $\mu$  then this breaks up as

$$\begin{aligned} \text{LHS} &= \left| \prod_{\mu} (\lambda\mu - 1) (\lambda^{-1}\mu - 1) (\lambda\mu^{-1} - 1) (\lambda^{-1}\mu^{-1} - 1) \right|^{1/2} \\ &= \left| \prod_{\mu} (\lambda^{1/2}\mu^{1/2} - \lambda^{-1/2}\mu^{-1/2}) (\lambda^{-1/2}\mu^{1/2} - \lambda^{1/2}\mu^{-1/2}) \right| \\ &= \left| \prod_{\mu} (\lambda + \lambda^{-1} - \mu - \mu^{-1}) \right| = \text{RHS}. \quad \square \end{aligned}$$



**Lemma 3.2.** *A basis of representatives for  $\Lambda/(wU - 1)\Lambda$  is given by*

$$(\sigma, 0), \quad \sigma \in \Lambda^R/(p - w - w^{-1})\Lambda^R.$$

**Proof:** By Lemma 3.1, these sets have the same number of elements. Now

$$(wU - 1)\Lambda = (U - w^{-1})\Lambda, \quad \text{where}$$

$$U - w^{-1} = \begin{bmatrix} a - w^{-1} & b \\ c & d - w^{-1} \end{bmatrix}.$$

As  $c = 1$ , there is clearly a basis of representatives of the form  $(\sigma, 0)$  (since there is an element of  $(U - w^{-1})\Lambda$  of the form  $(n, 1)$ ). Also,

$$\begin{bmatrix} a - w^{-1} & b \\ c & d - w^{-1} \end{bmatrix} \begin{bmatrix} -(d - w^{-1})\sigma \\ \sigma \end{bmatrix} = \begin{bmatrix} [p - w^{-1} - w]w^{-1}\sigma \\ 0 \end{bmatrix},$$

so for any  $\sigma \in \Lambda^R$ ,  $[(p - w - w^{-1})\sigma, 0] \in (wU - 1)\Lambda$ .  $\square$

**Remark:** As a set of representatives  $\lambda \in \Lambda$  can be chosen in this way, with the second component  $\lambda_2$  equal to zero, it is easy to see that the theta characteristics  $\epsilon(\lambda)$  can be chosen as 1.

We need the factor

$$L(A) = -2\pi \langle A_\lambda, S\lambda \rangle = 2\pi \langle A_2, \sigma \rangle,$$

where  $(wU - 1)A = \lambda$ . Explicitly,

$$\begin{bmatrix} (aw - 1) & bw \\ cw & (dw - 1) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \sigma \\ 0 \end{bmatrix},$$

so substituting for  $A_1$ , we obtain

$$A_2 = (w + w^{-1} - d - a)^{-1}\sigma.$$

Thus we have

**Proposition 3.3.** *The overall SQM formula for the mapping torus partition function is*

$$\begin{aligned} Z_{SQM}(U, k) &= i^\mu \frac{1}{|W|} \sum_{w \in W} \frac{\det(w)}{\sqrt{|\det(p - w - w^{-1})|}} \times \\ &\times \sum_{\sigma} \exp \left\{ -r\pi i \left\langle (p - w - w^{-1})^{-1}\sigma, \sigma \right\rangle \right\}. \quad \square \end{aligned}$$

Here the second sum is over  $\sigma \in \Lambda^R/(p - w - w^{-1})\Lambda^R$ , and  $\mu$  is the “defect” from [20] (see (68) in that paper).

#### 4. FRAMINGS OF MAPPING TORI

The Chern-Simons-Witten invariant of a 3-manifold  $Y$  depended on the specification of a *2-framing* (a trivialization of  $2TY$ ). For a discussion of how this dependence shows up in the path integral, see §5.4 in [20]. For mapping tori of surfaces  $\Sigma$ , the possible 2-framings correspond to maps from the mapping class group  $\Gamma$  to its  $\mathbb{Z}$ -central extension  $\hat{\Gamma}$  (see [5]). One such map  $s$  corresponds to the canonical framing (see §5.4 in [20]). If  $\Sigma$  is a torus, the extension actually splits, so another framing is defined by the unique homomorphism  $s_1 : \Gamma \rightarrow \hat{\Gamma}$ . The discrepancy between the framings  $s$  and  $s_1$  is discussed in [4], [5]: it is identified as the signature defect of  $\Sigma_U$  in the framing  $s_1$  (as the framing  $s$  is the one giving zero signature defect.) We have ([4], 6.15 and 5.4):

$$(15) \quad s_1(U) - s(U) = \Phi(U) - 3 \operatorname{sign} c(a + d),$$

where  $\Phi(U)$  is the Rademacher phi function [23]. (Actually, if  $U$  is hyperbolic, the signature defect of the mapping torus  $\Sigma_U$  is equal to its eta invariant in a natural metric: see [5]. In other words, in this case there is a canonical metric for which the “counterterm” vanishes.)

If  $U$  is hyperbolic, and conjugate to  $\operatorname{diag}(e^h, e^{-h})$ , then (see [4], (5.51)) the framing  $s_1$  on  $\Sigma_U$  can be constructed quite explicitly: it corresponds to using  $\operatorname{diag}(e^{th}, e^{-th})$  ( $0 \leq t \leq 1$ ) to define a path in the space of framings on  $\Sigma$ , and hence a framing on  $\Sigma_U$ . For simplicity, we restrict our comparison of phase factors to the case when  $U$  is hyperbolic (i.e., when  $\|\operatorname{Tr}(U)\| > 2$ .)

**Remark 4.1.** *Framing assumption:* Note that the symplectic quantum mechanics partition function contained an overall fourth root of unity  $i^\mu$  corresponding to the choice of a trivialization. *The formulas in this section are for the case when  $i^\mu = 1$ .*

4.1. *SU(2) case.* Apart from the factor  $i^\mu$  in (14), the discrepancy between (14) and (19) is a factor

$$iK\zeta^{\operatorname{sign} c(d+a\pm 2)}$$

multiplying the trace partition function. If  $U$  is hyperbolic, this is just  $i\zeta^{\operatorname{sign} c(d+a)-\Phi} \operatorname{sign} c$ .

Our discrepancy is

$$\zeta^{2-\psi-2 \operatorname{sign} c(d+a)+2 \operatorname{sign} c-2} = \operatorname{sign} (d+a)\zeta^{-\psi},$$

so we have

$$(16) \quad \operatorname{Tr}\mathcal{R}(U) = \zeta^{-\psi(U)} \operatorname{sign} (d+a) Z_{SQM}(\Sigma_U).$$

4.2. **General  $G$ .** The phase discrepancy between the SQM result (Proposition 3.3) and the result (21) for  $\operatorname{Tr}\mathcal{R}(U)$  is now

$$i^{|\Delta_+|} \exp\left\{-pi\pi \frac{|\rho|^2}{h}\right\} \exp i\pi l \frac{\operatorname{sign} \det(p-w-w^{-1})}{4}.$$

In the case when  $|p| > 2$ , this is

$$(17) \quad i^{|\Delta_+|} \exp\left\{-pi\pi \frac{|\rho|^2}{h}\right\} \exp i\pi l \frac{\text{sign } p}{4}.$$

Using  $|\Delta_+| = (\dim G - l)/2$ , this becomes

$$i^{|\Delta_+|} \exp\left\{-2\pi ip \frac{\dim G}{24}\right\} \exp i\pi \frac{l \text{ sign } p + \dim G - l}{4}.$$

In this case, the expected correction factor caused by the framing is

$$(18) \quad \exp\left\{-\frac{2\pi i \psi(U) \dim G}{24}\right\},$$

and  $\psi(U)$  in this case is  $p - 3 \text{ sign } p$ . A short calculation shows that equations (18) and (17) differ only by a sign  $(\text{sign } p)^{|\Delta_+|}$ .

Thus, up to a sign, the difference between the trace calculation and the SQM calculation for  $i^\mu = 1$  is accounted for by a change in framing, embodied in the factor  $\psi$ . The sign ambiguity is to be expected from the definition of the SQM partition function (see Remark 2.1), although we do not know how to resolve it.

**Remark:** Notice that once the phase has been corrected (by the procedure for regularizing the Chern-Simons theory eta invariant, described in §5.4 of [20]), one obtains *exact* agreement between the SQM result (or equivalently the stationary phase expansion of Chern-Simons: see §4 of [20]) and the result for  $\text{Tr} \mathcal{R}(U)$ . This is in contrast to the lens space case (see [19]), where one only obtains asymptotic agreement with the stationary phase formula.

## 5. GAUSS SUM DERIVATION OF CHERN-SIMONS PARTITION FUNCTION

5.1.  $G = SU(2)$ . Equation (4.7) of [19] reads as follows:

$$(19) \quad Z(U, r) = \sum_{\pm} \pm \frac{1}{2i|c|\sqrt{|d+a\pm 2|}} K(U) \zeta^{\text{sign}(c(d+a\pm 2))} \times \\ \times \sum_{\beta \pmod{c}} \sum_{\gamma=1}^{|d+a\pm 2|} \exp 2\pi i r \frac{-c\gamma^2 + (a-d)\gamma\beta + b\beta^2}{d+a\pm 2}.$$

Up to the phase which was the subject of §4, this is the SQM result (14). Note that the SQM result was expressed as a sum over a fundamental domain of  $\Lambda$  under the action of  $B = 1 \pm U$ . The equivalence of this with (14) is established by the following observations:

- (1)  $\det B = 2 \pm a \pm d$ .
- (2) The sum

$$(20) \quad \sum_{\beta=1}^{|c||d+a\pm 2|} \sum_{\gamma=1}^{|d+a\pm 2|} \exp 2\pi i r \frac{-c\gamma^2 + (a-d)\gamma\beta + b\beta^2}{d+a\pm 2}$$

equals  $||\det B||$  times the sum in (19). If  $(a, b)$  of  $\Lambda \cong \mathbb{Z}^2$  is such that  $\det B$  divides  $a$  and  $b$ , then  $(a, b)$  is in  $B\Lambda$ , so the points  $(\beta, \gamma) = (0, |d+a\pm 2|)$ ,  $(\beta, \gamma) =$

$(|c||d+a \pm 2|, 0)$  are in  $B\Lambda$ . Hence (20) covers precisely  $|\det B||c|$  fundamental domains (as it covers an integer number of domains, each of which contains  $|\det B|$  points).  $\square$

**5.2. General  $G$ .** The conjugacy classes of  $U$  for which  $c = 1$  can be represented by  $U = T^p S$ , where  $T$  and  $S$  are elements of  $SL(2, \mathbb{Z})$  which descend to the standard generators of  $PSL(2, \mathbb{Z})$  (see (11) and (12)). We obtain

$$(21) \quad Z = \text{Tr}(T^p S) = i^{|\Delta_+|} \left| \frac{\text{vol}(\Lambda^W)}{r \text{vol} \Lambda^R} \right|^{1/2} \exp \left\{ -\frac{pi\pi \langle \rho, \rho \rangle}{h} \right\} \times \\ \times \sum_{w \in W} \det(w) \sum_{\lambda} \exp \left\{ \frac{i\pi}{r} \langle (p - w - w^{-1})(\lambda + \rho), \lambda + \rho \rangle \right\},$$

where the sum is over  $\lambda \in \Lambda^W$  satisfying an integrality condition.

Let us analyse the symmetries of the trace sum (21) with a view to expressing it as a sum over  $\Lambda^W / r\Lambda^W$ . Define (for  $\lambda \in \Lambda^W$ )

$$g(\lambda) = \sum_{w \in W} \det(w) \cdot \exp \frac{\pi i \langle (p - 2w)\lambda, \lambda \rangle}{r}.$$

The trace is obtained by summing  $g(\lambda)$  over weights

$$\{\lambda = \mu + \rho : \mu \in \overline{\text{FWC}}, \langle \mu, \alpha_m \rangle \leq k\},$$

or

$$Z = \sum_{\lambda} g(\lambda)$$

where the sum is over  $\{\lambda \in \text{FWC} \mid \langle \lambda, \alpha_m \rangle < k + h.\}$

The following result is [19], Proposition 4.4:

**Proposition 5.1.**  *$g(\lambda)$  is invariant under:*

(i)  $\lambda \rightarrow -\lambda$  (obvious)

(ii)  $\lambda \rightarrow u\lambda, u \in W$  : for

$$\langle u\lambda, (p - 2w)u\lambda \rangle = \langle \lambda, \{p - 2(u^{-1}wu)\} \lambda \rangle.$$

(iii)  $\lambda \rightarrow \lambda + rh_\alpha, \alpha$  any root ( $h_\alpha$  denotes the corresponding coroot  $2\alpha/\langle \alpha, \alpha \rangle$ ). For

$$\frac{1}{r} \left\langle \lambda + rh_\alpha, (p - 2w)(\lambda + rh_\alpha) \right\rangle \\ = \frac{1}{r} \langle \lambda, (p - 2w)\lambda \rangle + 2 \langle h_\alpha, (p - 2w)\lambda \rangle + r \langle h_\alpha, (p - 2w)h_\alpha \rangle.$$

The second term is obviously in  $2\mathbb{Z}$ , since  $h_\alpha$  is in the integer lattice. The third term is also in  $2\mathbb{Z}$ , since  $\langle h_\alpha, h_\alpha \rangle \in 2\mathbb{Z}$  (a property of the basic inner product), and

$$\langle h_\alpha, wh_\alpha \rangle = \frac{2 \langle w\alpha, h_\alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

(iv)  $g(\lambda) = 0$  for a weight  $\lambda$  with  $\langle \lambda, \alpha \rangle = r n$  for any root  $\alpha$  ( $n \in \mathbb{Z}$ ).

The following result is Proposition 4.5 in [19]:

$$(22) \quad \text{Tr } \mathcal{R}(U) = \exp \left\{ -\frac{p\pi i \langle \rho, \rho \rangle}{h} \right\} i^{|\Delta_+|} \exp \left\{ \frac{i\pi l \text{sgn det}(B)}{4} \right\} \frac{|\text{det}(B)|^{-1/2}}{|W|} \times \\ \times \sum_{w \in W} \text{det}(w) \sum_{\mu \in \Lambda^{\mathbb{R}}/B\Lambda^{\mathbb{R}}} \exp -i\pi \langle \mu, rB^{-1}\mu \rangle.$$

The last expression equals what we obtained (Proposition 3.3) from the fixed point calculation, up to the phase which we investigated in the previous section.

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